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SOME ASPECTS OF MODULE THEORETIC HOMOMORPHISMS

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ABSTRACT

Here we try to highlight some aspects of module homomorphisms when the attached rings are different and hence, the already known cases of module homomorphisms come up as in some sense as corollaries to what we have proposed. This very approach may open up new windows so far the module homomorphisms are concerned (at least in some sense -the so called categorical approach is concerned). We would like to explore some interesting phenomena of module homomorphisms in some broader aspects. Moreover, it would be interesting to note how already available module theoretic results with same ring may have more than one version.

KEYWORDS: Homomorphisms, Module Theoretic, Injective Modules

1. INTRODUCTION

This paper is an attempt for some sort of generalization of modules and its homorphisms [1, 3, and 4] with different rings –best suited for some aspects of categorical study of rings and homomorphisms. As it appears as a first step towards the expected theory, we would like to present here some very familiar aspects of what we are attempting for. And in a natural approach we, here present, as most beginning to it the main aspect of the theory, viz., the homomorphism theorems etc with examples.

Here we would like to comprise some torsion character of modules with different rings, with some direct sum related problems. We see that homomorphic image of such a left module epimorphism (in broader aspect) appears with invariant character in the corresponding torsion part of the codomain module [4]. Moreover, some necessary and sufficient condition type property is carried over in case of integral domain makeup of the codomain module. It is also observed that simple character is also preserved in such type of left module morphisms. We here skillfully show the isomorphism character in case of quotient structure with well-behaved nature. If two left module morphisms agree on the generating set of a submodule of the codomain module, then the two left module morphisms agree also on the codomain submodule [2,3]. Finally we try to highlight some categorical property so far the kernel is concerned in such type of left module morphisms.

2. PRELIMINARIES

We begin with some definitions:

2.1 Definition

A ***left ring module*** is a pair (R, M), where *R* is a ring; *M* is an additive abelian group with

- $r(x+y)=rx+ry$
- $(r_1+r_2)x=r_1x+r_2x$

- $(r_1r_2)x=r_1(r_2x)$, where $r,r_1,r_2 \in R$ and $x,y\in M$.

Let (R_1,M_1) and (R_2,M_2) be two left ring modules. Then

2.2 A Left Module-Morphism

(α, f) is a system $(R_1,M_1)\xrightarrow{(\alpha,f)}(R_2,M_2)$ where $f:M_1\to M_2$ is an additive group homomorphism and $\alpha:R_1\to R_2$ is a ring homomorphism with

$f(r_1m_1)= \alpha(r_1)f(m_1)$, with $r_1 \in R_1$, $m_1\in M_1$

When the ring R_1 is considered as a left R_1- module, and a ring homomorphism $f:R_1\to R_2$ is such that for $\alpha:R_1\to R_2$ with $\alpha(r)= f(r)$, then we have $f(rs)=f(r)f(s)= \alpha(r)f(s)$. Thus $f(rs)= \alpha(r)f(s)$ and this ring homomorphism is a left module-morphism

$(R_1,R_1)\xrightarrow{(f,f)}(R_2,R_2)$ too.

The left module morphism (α, f) under consideration may also be called

2.3 "$f:M_1\to M_2$" is an (α,R_1-R_2) *homomorphism.* [so the left module morphism $(R_1,R_1)\xrightarrow{(f,f)}(R_2,R_2)$ is an $(f\text{-}R_1\text{-}R_2)$ homomorphism

Note: If $R_1 = R_2 = R$ then we get the left module morphism, $(R,M_1)\xrightarrow{(i,f)}(R,M_2)$ - the (i, R-R) homomorphism as our well known module homomorphism viz., R-homomorphism. For here, $f:M_1\to M_2$, where $i:R\to R$, with $i(r)=r$ $f(a_1+a_2)=f(a_1)+f(a_2)$ and $i(ra_1)= i(r)f(a_1)= rf(a_1)$, for $r, \in R$ and $a_1,a_2\in M$

2.4 A left module morphism $(R_1,M_1)\xrightarrow{(\alpha,f)}(R_2,M_2)$ is an *α- monomorphism*, (*α-epimorphism, α- isomorphism*) if $f:M_1\to M_2$ is injective(, surjective, bijective). If $(\alpha, f):(R_1,M_1):\to(R_2,M_2)$ is an *α- isomorphism,* we denote it by $(R_1,M_1)\cong_{\alpha}^{f}(R_2,M_2)$

An R_2-submodule N_2 of M_2 is an (α,R_2) *-submodule* of M_2 [denoted: $N_2\overset{\alpha(S_1)}{\leq}M_2$] if N_2 is an $\alpha(S_1)$ -submodule of M_2 for some subring S_1 of R_1. If α is onto, then $R_2= \alpha(R_1)$ thus, M_2 is an (α,R_2) -submodule of M_2 ;this M_2 , is then an

2.5 (α,R_2) *- module.*

2.6 Lemma

- If $N_1\overset{R_1}{\leq}M_1$ then $f(N_1)\overset{\alpha(R_1)}{\leq}M_2$, and

- For $A_2 \overset{R_2}{\leq} M_2$ we have $f^{-1}(A_2) \overset{R_1}{\leq}_{\alpha} M_1$ [it is to be noted the difference between $N_1 \overset{R_1}{\leq} M_1$ and $f^{-1}(A_2) \overset{R_1}{\leq}_{\alpha} M_1$]

Proof

$f(N_1)=\{f(n_1)|n_1 \in N_1\}$. Now, $\alpha(r_1) \in \alpha(R_1)$, $f(n_1) \in f(N_1)$ give $\alpha(r_1)\, f(n_1)= f(r_1n_1)=f(t_1) \in M_2$, [for, $t_1=r_1m_1 \in M_1$]

- Thus $f(N_1) \overset{\alpha(R_1)}{\leq} M_2$ //

And $f^{-1}(A_2)=\{m_1 \in M_1 \mid f(m_1) \in A_2\}$. Now, suppose, $r_1 \in R_1$, $m_1,m_2 \in f^{-1}(A_2)$. Then $f(m_1)$, $f(n_1) \in A_2$ and $f(m_1)-f(n_1)=f(m_1-n_1) =f(t^1) \in A_2$, as A_2 is an R_2-submodule of A_2. And, $f(r_1m_1)= \alpha(r_1)f(m_1) \in A_2$ gives, $r_1m_1 \in f^{-1}(A_2)$

and hence, $f^{-1}(A_2) \overset{R_1}{\leq}_{\alpha} M_1$ Here we know that our familiar kernel, kerf or, $f^{-1}(0_2)$ is an R_1 -submodule of M_1

Now, in case of $(\alpha, f):(R_1,M_1):\rightarrow(R_2,M_2)$

if $a,b \in \ker_{\alpha} f$, then $f(a_1)-f(b_1)= 0_2 \Rightarrow f(a_1-b_1)= 0_2$ [$a_1-b_1 \in M_1$]

so $a_1-b_1 \in \ker_{\alpha} f$ And, if $r_1 \in R_1$, $m_1 \in \ker_{\alpha} f$, then, $f(r_1m_1)=\alpha(r_1)f(m_1)= 0_2 \Rightarrow f(r_1m_1)= 0_2 \Rightarrow r_1m_1 \in \ker_{\alpha} f$. Thus, $r_1 \in R_1$, $m_1 \in \ker_{\alpha} f \Rightarrow r_1m_1 \in \ker_{\alpha} f$, Hence, $f^{-1}(0_2)$ or $\ker_{\alpha} f$ or, $f^{-1}(0_2) \overset{R_1}{\leq}_{\alpha} M_1$

i.e. $f^{-1}(0_2)$ is an (α,R_1)-submodule of M_1.[thus, $f^{-1}(0_2) \overset{R_1}{\leq}_{\alpha} M_1$]. $(R_1,M_1) \xrightarrow{(\alpha,f)} (R_2,M_2)$ is a left module morphism, i.e., $f:M_1 \rightarrow M_2$ is an $(\alpha, R_1 - R_2)$ homomorphism. So $f:M_1 \rightarrow M_2$ and $\alpha : R_1 \rightarrow R_2$ are as described.

Now,

2.7 Definition

α-kernel of f [or, kernel (α, f)] or, $K_{\alpha} =K_{\alpha}(f)= \{m_1 \in M_1|f(m_1)=0_2\}= f^{-1}(0_2)$

When $R_1=R_2=R$ and i, the identity homomorphism then, *α-kernel of f* is kernel of f, our familiar kernel of f viz., $\ker_{\alpha} f=\{m_1 \in M_1 \mid f(m_1)=0_2\}$. If $(A,+)$ is a sub group of $(M_2,+)$, then $f^{-1}(A) = \{ m_1 \in M_1 \mid f(m_1) \in A\}$ is an (α,R_1)-submodule of M_1

2.8 Definition

Let **C** be a class for each pair $A,B \in$ **C** we have $mor_c(A,B)$, a set . Elements of $mor_c(A,B)$ are "arrows" $f:A \rightarrow B$. A is called its domain and B its codomain. If $A,B,\ C \in$ **C** then we have a function $\circ : mor_c(B,C) \times mor_c(A,B) \rightarrow mor_c(A,C)$ such that for $g:B \rightarrow C \in mor_c(B,C)$ and $f:A \rightarrow B \in mor_c(A,B)$ we have $g \circ f : A \rightarrow C \in mor_c(A,C)$. If the system C= (**C**,mor_c,$\circ$) is such that C1. For every triple $h:C \rightarrow D$, $g:B \rightarrow C$, $f:A \rightarrow B$, $h \circ (g \circ f) = (h \circ g) \circ f$ C2. For each $A \in$ **C**, there is a unique

$1_A \in mor_c(A, A)$ such that if $f : A \to B$, $g : C \to A$, then $f \circ 1_A = f$ and $1_A \circ g = g$, then **C** is a *category*. Each arrow is a *morphism*.

2.9 Definition

A morphism $f : A \to B$ or $A \xrightarrow{f} B$ has *a kernel* if there is a morphism $M \xrightarrow{\alpha} A$ we have $f \circ \alpha = 0$ - the zero morphism, and for any morphism $C \xrightarrow{g} A$ with $f \circ g = 0$ - the zero morphism, we get a unique morphism $C \xrightarrow{h} M$ such that $\alpha \circ h = g$.

3. MAIN RESULTS

3.1 Theorem

Fundamental Theorem of α-Homomorphism

(R_1,M_1) and (R_2,M_2) are two left ring modules and $(R_1, M_1) \xrightarrow{(\alpha, f)} (R_2, M_2)$ is an left module- morphism [or, f is an α-R_1-R_2 homomorphism from M_1 to M_2, then $(R_1, M_1/\ker_\alpha f) \cong^{\phi}_{\alpha} (R_2, f(M_1))$ with ϕ: $M_1/\ker_\alpha f \to f(M_1)$ such that ϕ(a_1+$\ker_\alpha$f)= f(a_1) [$a_1 \in M_1$]

Proof: Here we have two mappings, viz., $\alpha : R_1 \to R_2$, with $r_1 \to \alpha(r_1)$ and $f : M_1 \to M_2$, with $m_1 \to f(m_1)$. We define a mapping ϕ: $M_1/\ker_\alpha f \to f(M_1)$ by ϕ(a_1+$\ker_\alpha$f)= f(a_1) [$a_1 \in M_1$]

- Φ is well defined. For, $a_1+\ker_\alpha f=b_1+\ker_\alpha f \Rightarrow a_1-b_1 \in \ker_\alpha f \Rightarrow f(a_1-b_1)=0 \Rightarrow f(a_1)= f(b_1)$ [f is additive group homo]
- Φ is one-one , $\phi(\overline{a_1}) = \phi(\overline{b_1})$ [i.e. $\phi(a_1+\ker_\alpha f)= \phi(b_1+\ker_\alpha f)$] $\Rightarrow f(a_1)= f(b_1) \Rightarrow f(a_1)- f(b_1)=0 \Rightarrow f(a_1- b_1)=0 \Rightarrow a_1- b_1 \in \ker_\alpha f \Rightarrow a_1+\ker_\alpha f= b_1+\ker_\alpha \Rightarrow \overline{a_1} = \overline{b_1}$
- Φ is onto-easy!

Now we see that Φ is an (R_1,R_2) –α homomorphism . Here, $\Phi(\overline{a_1} + \overline{b_1}) = \Phi(\overline{a_1 + b_1}) = f(a_1+b_1) = f(a_1)+f(b_1) = \phi(\overline{a_1}) + \phi(\overline{b_1})$ and $\Phi(r_1 \overline{a_1}) = \Phi(\overline{r_1 a_1}) = f(r_1a_1) = \alpha(r_1)\, f(a_1) = \alpha(r_1)\, \phi(\overline{a_1})$. Hence, ϕ: $M_1/\ker_\alpha f \to f(M_1)$ is an (α-R_1,R_2) isomorphism, i.e., $(R_1, M_1/\ker_\alpha f) \cong^{\phi}_{\alpha} (R_2, f(M_1))$ If $f : M_1 \to M_2$ is an epimorphism, then $f(M_1) = M_2$ And then $(R_1, M_1/\ker_\alpha f) \cong^{\phi}_{\alpha} (R_2, M_2)$ //

3.2 Theorem

Correspondence Theorem for Left Module Morphism

(R_1,M_1) and (R_2,M_2) are two left ring modules. If $(R_1,M_1)\xrightarrow{(\alpha,f)}(R_2,M_2)$ is a left module epimorphism, then there is a one-one correspondence between the set of all R_2-submodules A_2 of M_2 and those R_1-submodules of A_1 such that $\ker_\alpha f\subseteq A_1$ and; where, in other words the correspondence is of type: $(\ker_\alpha f\subseteq)A_1 = f^{-1}(A_2)\leftrightarrow A_2$

Proof: Here $(R_1,M_1)\xrightarrow{(\alpha,f)}(R_2,M_2)$ is a left-module epimorphism [or, (α-(R_1-R_2) epimorphism. i.e., $f: M_1\to M_2$ is an additive group epimorphism with the attached homomorphism $\alpha: R_1\to R_2$ as $r_1\to\alpha(r_1)$. And A_2 is an R_2-submodule of M_2. We take $f^{-1}(A_2)=A_1$ Here $\ker_\alpha f\subseteq f^{-1}(A_2)\overset{R_1}{\leq} M_1$ For $x_1,y_1\in f^{-1}(A_2)\Rightarrow f(x_1), f(y_1)\in A_2\Rightarrow$ $f(x_1-y_1)\in A_2\Rightarrow x_1-y_1\in f^{-1}(A_2)$. For $x_1\in f^{-1}(A_2)$, $r_1\in R_1$, we have $f(x_1)\in A_2,\alpha(r_1)\in R_2\Rightarrow\alpha(r_1)f(x_1)\in A_2\Rightarrow f(r_1x_1)\in A_2\Rightarrow$ $r_1x_1\in f^{-1}(A_2)$ Therefore, $f^{-1}(A_2)=A_1$ is an R_1-submodule of M_1 And clearly the correspondence is one-one- for if I_1 and J_1 are two R_1-submodules of M_1 with $\ker_\alpha f\subseteq I_1,J_1$ with $f(I_1)=f(J_1)$, then $I_1= f^{-1}(f(I_1))$ and $J_1= f^{-1}(f(J_1))$.//

Consider the map from the set of R_1-submodules of M_1 to the set R_2-submodules of $f(M_1)=M_2$, i.e.,

$$\beta:\left\{A_2 \mid A_2\overset{R_2}{\leq}M_2\right\}\to\left\{f^{-1}(A_2)(\overset{R_1}{\leq}M_1) \mid A_2\overset{R_2}{\leq}M_2\right\}$$,[β is one-one]

3.3 Example

$$Z[x]=\{a_0+a_1x+a_2x^2+...+a_nx^n \mid a_n\in Z, n=0,1,2..\},\ Z=\{o,\pm1,\pm2,\pm3,...\}$$

$$Z_2[x]=\{\bar{0},x,x^2,x+x^2,x^3,x+x^3,x+x^2+x^3,.....\}\ Z_2=\{\bar{0},\bar{1}\}$$

Here $Z[x]$ is a left Z -module. Similarly $Z_2[x]$ is a left Z_2 -module w.r.t. the maps .

$$Z\times Z[x]\to Z[x]:$$

$$(n,(a_0+a_1x+a_2x^2))\to(na_0)+(na_1)x+(na_2)x^2$$

$$Z_2\times Z_2[x]\to Z_2[x]:$$

$$(\bar{n},(\overline{a_0}+\overline{a_1}x+\overline{a_2}x^2))\to(\overline{na_0})+(\overline{na_1})x+(\overline{na_2})x^2$$ [In Z_2 $\bar{n}$ is either $\bar{0}$ or $\bar{1}$]

We consider the maps: $f: Z[x]\to Z_2[x]:$ $a_0+a_1x+a_2x^2\to\overline{a_0}+\overline{a_1}x+\overline{a_2}x^2$ i.e., $f(a_0+a_1x+a_2x^2)=\overline{a_0}+\overline{a_1}x+\overline{a_2}x^2$ and $\alpha: Z\to Z_2,\ n\to\bar{n}$ i.e. $\alpha(n)=\bar{n}$

Now $f(n(a_0+a_1x+a_2x^2)) = f((na_0)+(na_1)x+(na_2)x^2)$

$$=\overline{na_0}+\overline{na_1}x+\overline{na_2}x^2=\overline{na_0}+\overline{na_1}x+\overline{n_2}\,\overline{a_2}x^2=\overline{n}(\overline{a_0}+\overline{a_1}x+\overline{a_2}x^2)=\alpha(n)f(a_0+a_1x+a_2x^2)$$

Thus $f(n(a_0+a_1x+a_2x^2))=\alpha(n)f(a_0+a_1x+a_2x^2)$ Hence, f is an $\alpha, Z-Z_2$ morphism from, ${}_Z Z[x]$ to ${}_{Z_2}Z_2[x]$ $(Z,Z[x])\xrightarrow{(\alpha,f)}(Z_2,Z_2[x])$ is a left module morphism with $f:Z[x]\to Z_2[x]$ and $\alpha: Z\to Z_2$, as defined above.

3.4 Direct Sum of Modules over Different Rings

M_1 is a left R_1-module and R_2 is a left R_2-module. Now $M_1\oplus M_2=\{(m_1,m_2)\mid m_i\in M_i\}$ and $R_1\oplus R_2=\{(r_1,r_2)\mid r_i\in R_i\}$ It is easy to see that $R=R_1\oplus R_2$ is a ring and $M=M_1\oplus M_2$ is a left R-module it is not difficult to see that $M=M_1\oplus M_2$ is an additive abelian group. Now we define the map $R\times M\to M$ as $(r,m)\to rm$, where, $m=(m_1,m_2)$ and $r=(r_1,r_2)$, $rm=(r_1,r_2)(m_1,m_2)=(r_1m_1,r_2m_2)$ and it appears that M is a left R-module .And as ring $R_2\cong 0\oplus R_2$, $R_1\cong R_1\oplus 0$. Thus in this sense, as ring we can consider each of R_1 and R_2 as a subring of $R=R_1\oplus R_2$ and so $M=M_1\oplus M_2$ is also an R_1-module as well as a left R_2-module. Here for $m=(m_1,m_2)\in M$ and $r_1\in R_1$ gives $r=(r_1,0)\in R_1\oplus R_2$ and so $rm=(r_1,0)(m_1,m_2)=(r_1m_1,0)$ It is clear that M is a left $R_1\oplus 0$ –module i.e., M is a left R_1-module. Now M is a left R-module and M_1 is a left R_1-module. We define f: $M\to M_1$ such that $f(m)=f(m_1,m_2)=m_1$ Clearly, the map is onto and we see the map $\phi:R\to R_1$ where $\phi(r)=\phi(r_1,r_2)=r_1$ is also an onto map.

Now we see that f is an R-R_1 homomorphism.

For 1) $f(x+y)=f((x_1,x_2)+(y_1,y_2)) = f(x_1+y_1,x_2+y_2)=x_1+y_1=f(x_1,x_2)+f(y_1,y_2)=f(x)+f(y)$ For $r=(r_1,r_2)\in R$, $f(rx)=f[(r_1,r_2)(x_1,x_2)]=f(r_1x_1,r_2x_2)=f(r_1x_1)=r_1.x_1=\phi(r_1,r_2).f(x_1,x_2)=\phi(r)f(x)$

Thus $f(rx)=\phi(r)f(x)$ Clearly f is onto and one-one. And $\ker_\phi f=\{m\in M\mid f(m)=0_1\}=\{(m_1,m_2)\mid f(m_1,m_2)=0_1\}=\{(0,m_2)\mid m_2\in M_2\}(\cong M_2)$. So by fundamental theorem of homomorphism, we get:

3.5 Theorem

$$M_1\cong^f_\phi M/\ker_\phi f=(M_1\oplus M_2)/\ker_\phi f\cong (M_1\oplus M_2)/M_2 \quad\text{i.e.}\quad M_1\cong^f_\phi (M_1\oplus M_2)/M_2$$

$M_1\cong_\phi(M_1\oplus M_2)/M_2$. Similarly, $M_1\cong^f_\psi(M_1\oplus M_2)/M_2$, with $\psi(r)=f(r_1,r_2)=r_2$ We now note when a Ring modulo a one-sided ideal appears as a Quotient modules: Let (R_1,M_1) and (R_2,M_2) be two left ring modules. In other words, M_1 is a left R_1 –module and R_2 is a left R_2-module. Define $A_1=A(M_1)=\{r_1\in R_1\mid r_1m_1=0,\ m_1\in M_1\}$, then A_1 is an R_1-submodule of R_1 (i.e. it is a left ideal of R_1). For $r,s\in A_1\Rightarrow rm_1=0,\ sm_1=0\Rightarrow(r-s)m_1=0\Rightarrow r-s\in A_1$ for all $m_1\in M_1$. And for $r\in R_1$ and $r_1\in A_1$,

$(rr_1)m_1=r(r_1m_1)=r.0=0 \Rightarrow rr_1 \in A_1$, for all $m_1 \in M_1$ Similarly, $A_2=A(M_2)=\{r_2 \in R_2 | r_2m_2=0,\ m_2 \in M_2\}$ is an R_2-submodule of R_2 Thus, R_1/A_1 is an R_1-module similarly R_2/A_2 is an R_2-module. Then

3.6 Theorem

$\left(R_1, \frac{R_1}{A_1}\right) \cong_{\alpha}^{\phi} \left(R_2, \frac{R_2}{A_2}\right)$, for some epi $\alpha: R_1 \rightarrow R_2$. Define: ϕ: $R_1 \rightarrow R_2/A_2$ with $\phi\ (r_1)=\alpha(r_1)+A_2$ then for r_1 , $r,s \in R$, $\Phi(r+s)=\alpha(r+s)+A_2 = (\alpha(r)+ \alpha(s))+A_2= (\alpha(r) +A_2)+ (\alpha(s)+A_2)=\phi(r)+\phi\ (s)$ and, $\phi(rs)=\alpha(rs)+A_2=\alpha(r)\alpha(s)+A_2=\alpha\ (r)\ (\alpha(s) +A_2)=\alpha(r)\ \phi(s)$. ϕ is an epi, for $r_2+A_2 \in R_2/A_2, r_2 \in R_2$ and α being epi, we have $r_1 \in A_1$with $\alpha(r_1)=r_2$ thus $r_2+A_2= \alpha(r_1)+A_2= \phi\ (r_1)$, i.e. $\overline{r_2} = \phi(r_1)$ Therefore, ϕ is an α-R_1-R_2 epimorphism, or an α-epimomorphism. Now, by fundamental theorem,

$$\left(R_1, \frac{R_1}{A_1}\right) \cong_{\alpha}^{\phi} \left(R_2, \frac{R_2}{A_2}\right) //$$

3.7 Some More Results

(R_1, M) and (R_2, N) are two left ring modules. $(R_1, M) \xrightarrow{(\alpha,\phi)} (R_2, N)$ is a left module morphism. $[\phi: M \rightarrow N\ , \alpha: R_1 \rightarrow R_2\ ]$

Then $(\phi(Tor(_{R_1} M)) \subseteq Tor(_{R_2} N)$ with $Tor(_{R_1} M)$ = $\{m \in M | r_1m=0$, for some $r_1 \in R_1\}$

Proof: Let $\alpha: R_1 \rightarrow R_2$ be an epimorphism. Now we have, $Tor(_{R_1} M)$ = $\{m \in M | r_1m=0$, for some $r_1 \in R_1\}$

$\phi(Tor(_{R_1} M)) = \{\phi(m) \mid m \in Tor(M)\} = \{\phi(m) \mid \phi(r_1 m) = 0,\ for \quad r_1 \in R_1\}$

$= \{\phi(m) \mid \alpha(r_1)\phi(m) = 0,\ for \quad \alpha(r)_1 \in R_2\}$ $\Rightarrow \phi(m)[\in Tor(_{R_2} N)] \subseteq Tor(_{R_2} N)$ [as, $\phi(_{R_1} M) \subseteq {}_{R_2} N$

Thus, $\phi(Tor(_{R_1} M)) \subseteq Tor(_{R_2} N)$ .// **Note**: For (R_1, M) and (R_2, N) two left ring modules, if $(R_1, M) \xrightarrow{(\alpha,\phi)} (R_2, N)$ is a left module morphism, with $\phi: M \rightarrow N\ , \alpha: R_1 \rightarrow R_2$ .Then $\Phi(r_1x)+y)=\alpha(r_1)\ \phi(x)+\phi(y)$, for all $x,y \in M$ and for all $r_1 \in R_1$

Now

3.8 Theorem

We consider each of R_1 and R_2 is with unity**.** If R_2 is an integral domain with unity, then for a $\phi: M \rightarrow N\ , \alpha: R_1 \rightarrow R_2$, $(R_1, M) \xrightarrow{(\alpha,\phi)} (R_2, N)$ is a left module morphism if $\Phi\ (r_1x) +y) =\alpha\ (r_1)\ \phi(x) +\phi(y)$.

Proof: Taking r=1, we get $\Phi(x+y) = \phi(x) +\phi(y)$ [only when R_2 is an integral domain] And, for y=0, $\Phi\ (r_1x) =\alpha\ (r_1)\ \phi(x)$. Therefore, ϕ is an α -R_1-R_2 homomorphism. i.e., $(\alpha, \phi): (R_1, M) \rightarrow (R_2, N)$ is a left module morphism.//

3.9 Example

A left module morphism, α-epi[$f:M_1 \to M_2, \alpha: R_1 \to R_2$ If M_1 is simple, then f(M_1) is a simple R_2-submodule of *N* and if f(M_1)≠0 then f is one-one.

Proof: Here, clearly f(M_1) is an α(R_1) (if it is not epi) R_2-submodule of *N*-as α-epi. Now let T_2 be an R_2-submodule of f(M_1) Claim: $f^{-1}(T_2)$ is an R_1-submodule of M_1. Now, $f^{-1}(T_2)=\{m_1 \in M_1 \mid f(m_1) \in T_2\}$. For $m_1, n_1 \in f^{-1}(T_2) \Rightarrow f(m_1), f(n_1) \in T_2 \Rightarrow f(m_1-n_1) \in T_2 \Rightarrow m_1 - n_1 \in f^{-1}(T_2)$ And for $m_1 \in M_1$, $r_1 \in R_1$, $f(r_1m_1)=\alpha(r_1)f(m_1) \in T_2 \Rightarrow r_1m_1 \in f^{-1}(T_2) \Rightarrow f^{-1}(T_2)$ is an R_1-submodule of M_1. Since M_1 is simple, so either $f^{-1}(T_2)$ is 0 or M_1.Thus T_2=f(M_1) or, T_2=f(0)=0 ⇒ f(M_1) is simple [R_2-submodule of M_2]. If f(M_1) ≠0 then T_2≠0. We have, $f^{-1}(0) \subseteq f^{-1}(T_2)[=M_1$ or (0)] ⇒ f is a monomorphism.

3.10 Example

An R_1-submodule <S> of M_1 generated by a non-empty subset S_1 of M_1 $[S_1 \subseteq M_1]$ such that $<S>=\{\sum_i r_i x_i \mid r_i \in R, x_i \in S\}$. Consider two modules M_1, N_1 with morphism $(R_1, M_1) \xrightarrow{(\alpha,f)} (R_2, M_2)$ And $(R_1, M_1) \xrightarrow{(\alpha,g)} (R_2, M_2)$ If f(x)=g(x) for all x∈*S*, then f and g agree on the $\underline{R}_1$-submodule $<S>$. To show that f and g agree on $<S>$ i.e., f(s)=g(s) for all s∈*S* For s∈*S* , we have $r_1x_1 + r_2x_2 + ...r_tx_t$, t=1,2,… Now,

$$f(s) = f(r_1x_1 + r_2x_2 + ...r_tx_t) = f(r_1x_1) + f(r_2x_2) + ... + f(r_tx_t)$$

$$= \alpha(r_1)f(x_1) + \alpha(r_2)f(x_2) + ... + \alpha(r_t)f(x_t) = \alpha(r_1)g(x_1) + \alpha(r_2)g(x_2) + ... + \alpha(r_t)g(x_t)$$

$$= g(r_1x_1) + g(r_2x_2) + ... + g(r_tx_t) = g(r_1x_1 + r_2x_2 + ...r_tx_t) = g(s)$$

Therefore, f and g agree on the R_1-submodule $<S>$ //

3.11 Theorem

Let A_1 and A_2 be R_1 and R_2 submodules of M_1 and M_2 respectively. Then,

$$\left(R_1 \times R_2, \frac{M_1 \times M_2}{A_1 \times A_2}\right) \cong_{\alpha_1 \times \alpha_2} \left(R_1 \times R_2, \frac{M_1}{A_1} \times \frac{M_2}{A_2}\right)$$, where, $(R_1, M_1) \xrightarrow{(\nu_1, i_1)} (R_1, \frac{M_1}{A_1})$

and $(R_2, M_2) \xrightarrow{(\nu_2, i_2)} (R_2, \frac{M_2}{A_2})$ are left module morphisms with $\nu_1 : M_1 \to \frac{M_1}{A_1}$ and $\nu_2 : M_2 \to \frac{M_2}{A_2}$ as natural additive group epimorphisms and i_1, i_2 are respective identity homomorphisms with

$\alpha_1 : R_1 \to R_1$ and $\alpha_2 : R_2 \to R_2$ be any ring homomorphisms.,

Proof: Here, the corresponding homomorphisms are as follows: $M_1 \times M_2$ and $\dfrac{M_1}{A_1} \times \dfrac{M_2}{A_2}$ are both $R_1 \times R_2 (= R)$ -modules with rm= r $(m_1, m_2) = (r_1, r_2)(m_1, m_2) = (r_1 m_1, r_2 m_2)$ and

$= r(\overline{m_1}, \overline{m_2}) = (r_1, r_2)(\overline{m_1}, \overline{m_2}) = (\overline{r_1 m_1}, \overline{r_2 m_2})$

Define $(\nu =)\nu_1 \times \nu_2 : M_1 \times M_2 \to \dfrac{M_1}{A_1} \times \dfrac{M_2}{A_2}$ such that

$\nu(m) = (\nu_1 \times \nu_2)(m_1, m_2) = (\nu_1(m_1), \nu_2(m_2))[= (\overline{m_1}, \overline{m_2}) = (m_1 + A_1, m_2 + A_2)]$.

[where m=m_1 $\times m_2$] Clearly, $\nu = \nu_1 \times \nu_2$ is well defined as well as onto. Moreover it is $\alpha_1 \times \alpha_2$ [: $R_1 \times R_2 \to R_1 \times R_2$] homomorphism. For $\nu(rm) = (\nu_1 \times \nu_2)[(r_1, r_2)](m_1, m_2) = (\nu_1 \times \nu_2)(r_1 m_1, r_2 m_2)$ $= (\nu_1(r_1 m_1), \nu_2(r_2 m_2)) = (\alpha_1(r_1)\nu_1(m_1), \alpha_2(r_2)\nu_2(m_2))$

Here, $[= (\overline{m_1}, \overline{m_2}) = (m_1 + A_1, m_2 + A_2)]$ $(\alpha_1(r_1)\nu_1(m_1), \alpha_2(r_2)\nu_2(m_2)) = (\alpha_1 \times \alpha_2)(r_1 \times r_2)(\nu_1 \times \nu_2)(m_1, m_2) = \alpha(r)\nu(m)$

Thus, $\nu(rm) = \alpha(r)\nu(m)$ Now $\ker_\alpha \nu = \{(m_1, m_2) \mid \nu(m_1, m_2) = (\overline{0_1}, \overline{0_2}) = (A_1, A_2)\}$ $= \{(m_1, m_2) \mid (\overline{m_1}, \overline{m_2}) = (\overline{0_1}, \overline{0_2})\} = \{(m_1, m_2) \mid m_1 \in A_1, m_2 \in A_2\} = A_1 \times A_2$ And by fundamental theorem

$$\left(R_1 \times R_2, \frac{M_1 \times M_2}{A_1 \times A_2}\right) \cong^{(\nu \times \nu_2)}_{(\alpha_1 \times \alpha_2)} \left(R_1 \times R_2, \frac{M_1}{A_1} \times \frac{M_2}{A_2}\right) \text{ or}$$

1`

$$\left(R_1 \times R_2, \frac{M_1 \times M_2}{A_1 \times A_2}\right) \cong^{\nu}_{\alpha} \left(R_1 \times R_2, \frac{M_1}{A_1} \times \frac{M_2}{A_2}\right)$$

4. SOME CATEGORICAL OBSERVATION

4.1 Every left morphism has a kernel in categorical sense [1]. . $(R_1, M_1) \xrightarrow{(f,\alpha)} (R_2, M_2)$ is a left module morphism, where $f : M_1 \to M_2$ is group homomorphism and $\alpha : R_1 \to R_2$ is ring homomorphism. We write $M' = Kerf$ and $R' = Ker\alpha$ Then we have $(R', M') \xrightarrow{(i,j)} (R_1, M_1) \xrightarrow{(f,\alpha)} (R_2, M_2)$, $i : M' \to M$ and $j : R' \to R_1$ are natural inclusion maps such that $(f, a) \circ (i, j) = (f \circ i, a \circ j) = (0,0)$ We assume that $(R_3, M_3) \xrightarrow{(g,b)} (R_1, M_1) \xrightarrow{(f,\alpha)} (R_2, M_2)$ with $(f, a) \circ (g, b) = (0,0)$.

Then $(f, a) \circ (g, b) = (0,0) \Rightarrow (f \circ g, a \circ b) = (0,0) \Rightarrow f \circ g = 0, a \circ b = 0$ ------ (i) We consider $(R_3, M_3) \xrightarrow{(h,c)} (R', M')$ with h(m_3)=g(m_3) and c(r_3)=b(r_3) As $f(g(m_3)) = (f \circ g)(m_3) = 0 \Rightarrow g(m_3) \in M'$ and

$a(b(r_3)) = (a \circ b)(r_3) = 0 \Rightarrow b(r_3) \in R'$. Thus (h,c) is well defined. Now $g(m_3)=h(m_3)=i(h(m_3))= (i \circ h)(m_3)$ for all $m_3 \in M_3$. Thus $i \circ h = g$. Similarly, $j \circ c = b$. Therefore, $(g,b) = ((i \circ h, j \circ c) \Rightarrow (g,b) = (i, j) \circ (h,c)$. Also (h,c) is unique. Hence (f,a) has a kernel in categorical sense.//

Note: One may look into the possibility of some approach for another interesting isomorphism character as described below. However this is our suggestion only. If we have a left module morphism of the type $(R_1, M_1) \xrightarrow{(\alpha, f)} (R_2, M_2)$, where $f : M_1 \to M_2$ is an additive group homomorphism and $\alpha : R_1 \to R_2$ is a ring homomorphism with $f(r_1m_1)= \alpha(r_1)f(m_1)$, with $r_1 \in R_1$, $m_1 \in M_1$ Then we may give the following definitions: The left module morphism $(R_1, M_1) \xrightarrow{(\alpha, f)} (R_2, M_2)$ is an *f-isomorphism* if $\alpha : R_1 \to R_2$ is bijective. We denote it by $(R_1, M_1) \cong_f (R_2, M_2)$]

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